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Minkowskian description of polarized light and polarizers

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A conventional Stokes description of polarized light is considered in a four-dimensional Lorentzian space, developing a seminal idea of Paul Soleillet [Ann. Phys. (Paris) **12**, 23 (1929)]. This provides a striking interpretation for the degree of polarization and the Stokes decomposition of light beams. Malus's law and reciprocity theorems for polarizers are studied using this Lorentzian formalism.

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I. INTRODUCTION

Polarization phenomena are associated with transverse waves and, at sufficiently long distance from the sources (plane wave zone), they are described by four real parameters including total intensity. For example, a quasimonochromatic partially polarized plane light wave is traditionally represented as an incoherent superposition of an unpolarized light and a totally polarized one; then, its polarization state is specified by the intensity of its components and the geometry (orientation, eccentricity, and sense) of the polarization ellipse described by the electric field of the totally polarized component [1,2].

The main conceptual advances in the comprehension of polarization are achieved with the development of the optical coherence theory. From a statistical point of view, polarization is interpreted as the existence of a correlation between the fluctuations of two mutually perpendicular components of the electric field at a fixed space point [3,4]. When there is no such correlation, light is named unpolarized. Useful parameters extensively employed in electromagnetic polarization phenomena are the so called *Stokes parameters*, a set of physically measurable quantities from diverse procedures (see Refs. [4,5] for a brief discussion of several methods). Stokes parameters involve only second-order statistics of the (stationary and ergodic) stochastic process associated with an optical field and they provide a whole description of the field polarization properties although, in general, they do not determine higher-order statistics (non-Gaussian process).

There are several ways to represent polarized radiation fields, the Stokes description, the Poincaré (sphere) representation, and the Wolf coherency matrix formalism being the most popular ones [1]. The first and second are mainly used in relation to transfer equations for polarized light and in matrix methods in optics [2,5]. The third is also well adapted to complex representations, currently employed in quantum mechanics dealing with polarization phenomena for both massless and massive elementary particles [6]. But when the ordered 4-tuple of Stokes parameters is considered as a vector in a four-dimensional Lorentzian space, these representations are algebraically equivalent.

The possibility of using a Lorentzian metric to describe light polarization was already pointed out by Soleillet [7] but, to our knowledge, he did not develop this idea any further. Later, Perrin [8] employed a Minkowski four-dimensional space to analyze the algebraic structure of some scattering matrices and, more recently, several authors [9–12] have considered the matrix description of optical linear systems from this point of view. However, many other aspects of the Soleillet idea remain unexplored, and its development could provide a way of translating the Minkowskian language from relativistic physics to polarization phenomena [13]. Soleillet was also the first to introduce matrix methods in the study of the lineal interaction of polarized light with optical systems. This other pioneering aspect of the Soleillet work has recently been emphasized by Brosseau [4] claiming for Soleillet the corresponding rights in the creation of the so called *Mueller formalism* in optics. We also refer to the Brosseau's monography [4] in connection with the important role played nowadays by polarization phenomena in several branches of science and technology [14]. In this work, we expect to gain an insight into the other original Soleillet's idea of describing polarization in a geometric Minkowskian language. This is our main aim.

This paper is organized as follows. We begin in Sec. II by describing polarization in the language of Minkowskian geometry. Unpolarized or natural light is associated with a distinguished timelike future-pointing vector. Partially polarized light is represented by any other timelike vector of the same time orientation as natural light. Its degree of polarization is related with the hyperbolic angle between both timelike directions. A future-pointing null direction represents totally polarized radiation. These null vectors generate the Stokes null cone of this Lorentzian structure, which is termed Stokes space. In Sec. III, we consider polarizers from a Lorentzian point of view, interpreting their representative matrices as an homothetic Lorentz boost acting on Stokes space. In particular, two common examples (polarization on reflection and refraction, and polarization produced by Thomson scattering) are considered. In Sec. IV we present the law of reciprocity in optics for ideal polarizers using the Stokes space formalism and, lastly, an extension of this law for nonideal polarizers is obtained from a generalized Malus law. Finally, in Sec. V, we summarize the main results and comment on the physical interest of the Lorentzian approach in polarization phenomena. Some of these results were com-

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municated at the Spanish relativistic meeting, ERE 2001 [15].

II. LORENTZIAN DESCRIPTION OF POLARIZED LIGHT

Stokes parameters are defined for monochromatic plane waves, and then extended for quasimonochromatic ones whose amplitudes and phases are slowly varying functions on the scale of the coherence time. According to conventional notation, they are arranged in an ordered 4–tuple $S=(I,Q,U,V)$ subjected to two physical constraints [1],

$$I>0, \quad I^2 \geqslant Q^2+U^2+V^2, \tag{1}$$

I being the light intensity and where the equality occurs for totally polarized light. Parameters U and Q are related to the linear polarization and V gives information about circular polarization. The constraints mentioned above may be interpreted geometrically using a *Lorentzian* terminology; they represent the points that are *within* or *on* the positive shell of a Minkowskian cone (*Stokes cone*). The polarization degree P satisfies

$$0 \leqslant P \equiv \frac{\sqrt{Q^2+U^2+V^2}}{I} \leqslant 1. \tag{2}$$

Mathematically, the parameter I can be extended for any real value in order to consider the set of 4-tuples $\mathbf{S}=\{(I,Q,U,V)\}=\mathbb{R}^4$. This set, endowed with a Lorentzian metric will be called the (extended) *Stokes space*. Let $S=(I,Q,U,V)$ and $S'=(I',Q',U',V')$ be two *Stokes vectors*, that is, $S,S' \in \mathbf{S}$, then $(S,S')=II'-QQ'-UU'-VV'$ is their scalar product.

Completely polarized lights $(P=1)$ are represented by null vectors S, $(S,S)=0$, with positive intensity and they generate the *Stokes cone*. A completely unpolarized $(P=0)$ or *natural* light of intensity $I>0$ is represented by $S=Iu$, where $u=(1,0,0,0)$, $(u,u)=1$. A *partially* polarized light with $0<P<1$ is a positive oriented vector $S \notin \{u\} \equiv \{\lambda u, \lambda \in \mathbb{R}\}$ pointing into the Stokes cone, $(S,S)>0$ and $I>0$. Then, the intensity of (a light represented by) S is the scalar product of u and S, $I=(u,S)>0$. The set of null Stokes vectors corresponds to the set of two-component complex vectors of the Jones formalism (Jones' vectors), frequently employed in optics [16]. However, Jones formalism only applies to phenomena involving totally polarized beams. In contrast, when one deals with depolarization processes the use of both positive and null Stokes' vectors are required. Hence, the use of the Minkowskian geometry can be relevant in dealing with partially polarized light and changes of its degree of polarization. Furthermore, the role played by the Lorentz group in modeling the usual devices commonly employed in linear optics (e.g., polarizers and retarders) has been extensively analyzed in the literature [17].

Any partially polarized light S can be expressed as a linear combination of u and a null vector l. Geometrically, the intersection of the Stokes cone and the two-plane expanded by S and u gives two null directions $\{l\}$ and $\{m\}$ that represent totally polarized light beams with opposite polarizations. This decomposition of S is the formal interpretation of Stokes representation of partially polarized light as an incoherent mixture of natural and totally polarized light and it is expressed in the following way:

$$S=(I-I_p)u+l, \tag{3}$$

where l is a totally polarized light whose intensity is given by $I_p=(u,l)$. Another decomposition of S, with $I_p=(u,m)$, is obtained replacing l by m in Eq. (3).

Now, let us examine how the polarization degree can be interpreted in the Lorentzian approach to polarization phenomena. From Eq. (3), the norm of S is $(S,S)=I^2-I_p^2$ and from Eq. (2), the degree of polarization is expressed as

$$P=\frac{I_p}{I}=\sqrt{1-\frac{(S,S)}{(u,S)^2}}. \tag{4}$$

Introducing the unit vector s and using the familiar relativistic notation, we can write

$$S=\sqrt{(S,S)}s=\sqrt{(S,S)}\gamma(1,\vec{\beta})=I(u+\beta n), \tag{5}$$

where

$$\gamma=(u,s), \quad \vec{\beta}=\beta n=(\mathcal{q},\mathcal{u},\mathcal{v}), \tag{6}$$

with $(u,n)=0$ and $(n,n)=-1$, and $\mathcal{q}\equiv Q/I$, $\mathcal{u}\equiv U/I$, $\mathcal{v}\equiv V/I$ being the *normalized* Stokes parameters. Then, Eq. (4) is written as

$$P=\sqrt{1-\frac{1}{\gamma^2}}=\beta\equiv|\vec{\beta}| \tag{7}$$

that provides a new interpretation of the degree of polarization.

Proposition 1. In the Lorentzian representation of polarized radiation, the degree of polarization P is interpreted as the norm β of the "relative velocity" between the unitary Stokes vectors u and s associated, respectively, with natural and partially polarized light.

Furthermore, from Eq. (3) and setting $u=(l+m)/(2I_p)$, each polarized light S of intensity I can be decomposed according to the following expression:

$$S=\frac{1}{2\beta}[(1+\beta)l+(1-\beta)m], \tag{8}$$

where both l and m are null vectors which have the same intensity $I_p=(u,l)=(u,m)=\beta I$, and with opposite projections in the three-space orthogonal to u. The physical meaning of Eq. (8) is clear because it reflects the well known equivalence between a light beam having intensity I and polarization degree β, and two incoherent streams of totally polarized light having states of opposite polarization and intensities $(1+\beta)I/2$ and $(1-\beta)I/2$.

This interpretation is also extended to the decomposition of natural light of intensity I in two incoherent opposite polarized waves with the same intensity $I/2$. These waves can be linearly polarized and mutually perpendicular, or circularly polarized with opposite helicities, one right handed and the other left handed (cf. Refs. [1,2]). It is remarkable that natural light is the only one that is contained in every two-plane generated by two arbitrary opposite polarization states. Finally, let us observe that the two decompositions of natural light mentioned above give a qualitative interpretation of Zeeman's effect produced by magnetic fields which are, respectively, tranverse or longitudinal with respect to the line of sight.

III. LORENTZIAN INTERPRETATION OF POLARIZERS

Starting with the vector u representing unitary natural light, we can consider the two-tensor $T=u\otimes u$. For a given polarized light $S=I(u+\beta n)$, we have $T(S)=Iu$. So, T produces a pure depolarizing effect, without changing the intensity. By contrast, a polarizer produces the opposed effect, it gives a polarized light beam from an unpolarized one. The matrix polarizer representation that is usually employed in optics [5] is written in the following way using tensorial notation

$$A=\frac{1}{2}(k_1 l\otimes m+k_2 m\otimes l)+\sqrt{k_1k_2}h, \tag{9}$$

with k_1 and k_2 as the transmission coefficients ($0\leqslant k_2\leqslant k_1\leqslant 1$) and where

$$l=u+e, \quad m=u-e, \quad -h=p\otimes p+q\otimes q, \tag{10}$$

$\{u,e,p,q\}$ being an orthonormal basis of Stokes space. Algebraically, k_1 and k_2 are eigenvalues of A having associated null eigendirections $\{l\}$ and $\{m\}$, $A(l)=k_1 l$ and $A(m)=k_2 m$; the vectors l and m represent totally polarized lights with opposite polarizations. The special case of an *ideal* polarizer corresponds to $k_1=1$ and $k_2=0$ because such a device transmits light without absorption and only in a given direction. Therefore, an ideal polarizer can be interpreted as a projector on a null direction of Stokes space. In the generic case, let us denote

$$a=\sqrt{k_1k_2}\neq 0, \quad \psi=\frac{1}{2}\ln\frac{k_1}{k_2}, \tag{11}$$

and, looking for a suitable parametric Lorentzian form for polarizers, Eq. (9) can be expressed as $A=a\Lambda$, where Λ is given by

$$\Lambda=\cosh\psi(u\otimes u-e\otimes e)-\sinh\psi(u\otimes e-e\otimes u)+h. \tag{12}$$

This expression corresponds to the tensorial form of the matrix expression of a Lorentz transformation, that is, an hyperbolic rotation in the two-plane generated by u and e. This two-plane contains exactly two null directions which are the eigendirections of the polarizer $\{l\}$ and $\{n\}$. The 4×4 matrix form of Eq. (12) is more usually employed in applied optics, as can be found in some references quoted in Ref. [17]. Consequently, the linear transformation on Stokes space representing a nonideal polarizer can be seen as the composition of an homothetic transformation aI_d, where I_d is the 4×4 identity matrix, and a special Lorentz transformation Λ (boost on the $\{l,m\}\equiv\{u,e\}$ plane) whose velocity parameter is

$$b=\tanh\psi=\frac{k_1-k_2}{k_1+k_2}. \tag{13}$$

Note that this velocity can also be interpreted as the degree of polarization P that the polarizer produces when it acts on natural light according to the relation

$$A(u)=\frac{1}{2}(k_1 l+k_2 m)=\frac{1}{2}(k_1+k_2)\left[u+\frac{k_1-k_2}{k_1+k_2}e\right] =I(u+be), \tag{14}$$

where $e=(l-m)/2$ and $I=(k_1+k_2)/2$ is the intensity of the transmitted beam, when the incoming natural light is unitary. The inversion of Eqs. (11) and (13) gives the eigenvalues of A in terms of the polarizer parameters a and b,

$$k_1=a\sqrt{\frac{1+b}{1-b}}, \quad k_2=a\sqrt{\frac{1-b}{1+b}}. \tag{15}$$

From Eq. (9), the use of the tensorial notation will allow to obtain results in the following section about the generalized Malus law and reciprocity theorems in optics. Also, it will make possible to express these results in a more compact form than in the current matrix formalisms. Before that, let us gain more insight into the Lorentzian description of some usual polarization phenomena. As a first example, we specialize to light polarization on ordinary reflection and refraction [1]. Let us consider light traveling from a dielectric medium M_1 of refractive index n_1 to another one M_2 whose index is $n_2=nn_1$. The media are separated by a flat interface and are considered homogeneous and isotropic. Let $\mathcal{R}$ ($\mathcal{T}$) be the reflectivity (transmissivity), that is, the ratio of the amount of reflected (refracted) energy flux to the incident one. As a consequence of *Fresnel's formulas*, $\mathcal{R}$ and $\mathcal{T}$ depend on the incidence angle (θ_i), the refraction angle (θ_t), and the polarization state of the incident light. In particular, for perpendicular ($\perp$) and parallell ($\parallel$) polarizations with respect to the incidence plane one has the respective associated quantities $\mathcal{R}_\perp$, $\mathcal{R}_\parallel$, $\mathcal{T}_\perp$, and $\mathcal{T}_\parallel$, which only depend on θ_i and θ_t according to the following expressions [1]:

$$\mathcal{R}_\perp=\frac{\sin^2(\theta_i-\theta_t)}{\sin^2(\theta_i+\theta_t)}, \quad \mathcal{R}_\parallel=\frac{\tan^2(\theta_i-\theta_t)}{\tan^2(\theta_i+\theta_t)}, \qquad \mathcal{T}_\perp=\frac{\sin 2\theta_i \sin 2\theta_t}{\sin^2(\theta_i+\theta_t)}=\mathcal{T}_\parallel\cos^2(\theta_i-\theta_t). \tag{16}$$

We consider that the incidence angle is smaller than the critical angle that corresponds to total refraction. The linear polarization degrees $P_{\mathcal{R}}$ and $P_{\mathcal{T}}$ associated with the reflected and refracted waves (for incident natural light) are defined, respectively, by

$$P_{\mathcal{R}}=\left|\frac{\mathcal{R}_{\perp}-\mathcal{R}_{\parallel}}{\mathcal{R}_{\perp}+\mathcal{R}_{\parallel}}\right|,\quad P_{\mathcal{T}}=\left|\frac{\mathcal{T}_{\perp}-\mathcal{T}_{\parallel}}{\mathcal{T}_{\perp}+\mathcal{T}_{\parallel}}\right|. \tag{17}$$

So, for each incidence angle, the polarization on reflection can be described by Eq. (9) with transmission coefficients $k_1=\mathcal{R}_{\perp}\geqslant\mathcal{R}_{\parallel}=k_2$, when $n_1\leqslant n_2$. And, in this case, the polarization on refraction is described in a similar way considering $k_1=\mathcal{T}_{\parallel}\geqslant\mathcal{T}_{\perp}=k_2$. The matrix forms corresponding to these situations can be found in Refs. [4,14] and references therein.

An outstanding situation occurs for Brewster incidence angle, which is given by the condition $\theta_i+\theta_t=\pi/2$ and corresponds to $\tan\theta_i=n$. At this angle, we have $\mathcal{R}_{\parallel}=0$, $\mathcal{T}_{\parallel}=1$, and

$$\mathcal{R}_{\perp}=\cos^2(2\theta_i)=\left(\frac{1-n^2}{1+n^2}\right)^2,\quad \mathcal{T}_{\perp}=\sin^2(2\theta_i)=\left(\frac{2n}{1+n^2}\right)^2. \tag{18}$$

Consequently, at Brewster angle, the polarization on reflection is described by a quasiideal polarizer with $k_1=\mathcal{R}_{\perp}$ and $k_2=0$. Moreover, for the refracted light, when a new refraction from M_2 to M_1 is produced, we can write

$$k_1=\mathcal{T}_{\parallel}\mathcal{T}'_{\parallel}=1,\quad k_2=\mathcal{T}_{\perp}\mathcal{T}'_{\perp}=\left(\frac{2n}{1+n^2}\right)^4, \tag{19}$$

$\mathcal{T}(\mathcal{T}')$ being the transmissivity from M_1 to M_2 (from M_2 to M_1). Note that $\mathcal{T}_{\parallel}=\mathcal{T}'_{\parallel}$ and $\mathcal{T}_{\perp}=\mathcal{T}'_{\perp}$ at Brewster angle. From Eq. (11) the homothetic and boost parameters are now given by

$$a=\left(\frac{2n}{1+n^2}\right)^2,\quad \psi=2\ln\left(\frac{1+n^2}{2n}\right).$$

When repeated refractions are produced using a pile of d identical polarizer plates, the polarization degree of the transmitted light is obtained taking into account Eq. (13), and it results

$$b=\tanh\psi=\frac{1-\left(\frac{2n}{1+n^2}\right)^{4d}}{1+\left(\frac{2n}{1+n^2}\right)^{4d}}.$$

Finally, another example for consideration is the matrix associated with a process of Thomson scattering of photons by free electrons, whose expression is [6]

$$A(\theta)=\frac{1}{2}r_0^2\begin{pmatrix}1+\cos^2\theta & \sin^2\theta & 0 & 0\\ \sin^2\theta & 1+\cos^2\theta & 0 & 0\\ 0 & 0 & 2\cos\theta & 0\\ 0 & 0 & 0 & 2\cos\theta\end{pmatrix}, \tag{20}$$

where $r_0=e^2/mc^2=2.82\times10^{-13}$ cm is the classical electron radius and θ the scattering angle (in the laboratory rest frame). From a Lorentzian point of view, this matrix represents a linear polarizer with

$$a(\theta)=r_0^2\cos\theta,\quad b(\theta)=\tanh\psi(\theta)=\frac{\sin^2\theta}{1+\cos^2\theta}.$$

In this case, the transmission coefficients are $k_1=k_{\perp}=r_0^2$ and $k_2=k_{\parallel}=r_0^2\cos^2\theta$ and now, the subscripts represent the perpendicular and parallel projections with respect to the scattering plane. Their ratio is $1:\cos^2\theta$, according to the Rayleigh's law [2]. For $\theta=\pi/2$ the radiation is totally polarized and the direction of the electric field is perpendicular to the scattering plane.

IV. MALUS'S LAW AND RECIPROCITY THEOREMS

In this section, we consider the *reciprocity law* in optics and some of its extensions using Stokes space description. For this and related issues we refer to a work by Perrin [8], where some interesting comments about its range of validity and its connection with quantum principles can be found.

For the sake of clarity, we study first the case of an ideal polarizer. It can be represented in the following tensorial form:

$$N=\frac{1}{2}l\otimes m=\frac{1}{2}(u\otimes u-e\otimes e-u\otimes e+e\otimes u), \tag{21}$$

as can be seen from Eq. (9) with $k_1=1$, $k_2=0$, $l=u+e$, and $m=u-e$. Its action on a Stokes vector $S=I(u+\beta n)$ gives

$$N(S)=\frac{1}{2}(m,S)l=\frac{I}{2}[1-\beta(e,n)]l. \tag{22}$$

Contracting this expression with u, we obtain the light intensity after crossing the polarizer.

Proposition 2. When a light beam of intensity I_{in} and degree of polarization β crosses an ideal polarizer, the intensity of the outgoing beam is given by

$$I_{out}=\frac{1}{2}[1-\beta(e,n)]I_{in}, \tag{23}$$

where e and n represent the unitary orthogonal part (relative to the unitary natural light u) of the Stokes vector associated, respectively, with the outgoing and incoming beams.

Notice that the *Malus law* is recovered from Eq. (23) when the incoming light is completely polarized ($\beta=1$) and taking $(e,n)=-\cos2\theta$, according the chosen Lorentzian signature of Stokes space. Then, $I_{out}=I_{in}\cos^2\theta$. When the incoming beam is linearly polarized, θ is exactly the angle between the incident electric field and the polarizer transmission axis. Observe that a rotation of angle α around the direction of light propagation corresponds to a rotation of angle 2α in the $\{Q,U\}$ plane of Stokes space. Moreover, taking into account Proposition 2, we can analyze the reciprocity law for an optical system constitued by two ideal polarizers represented as N and $N'=\frac{1}{2}l'\otimes m'$, with $l'=u+e'$ and $m'=u-e'$. Contracting with u the matrix product NN', we find

$$4i(u)NN'=(m,l')m'=[1-(e,e')]m', \tag{24}$$

where $i(x)$ stands for the inner product with the vector x, $[i(x)N]_{\nu}=x^{\mu}N_{\mu\nu}$. So, from Eqs. (24) and (22) and using the fact that $2i(u)N(S)=(m,S)$, we arrive at the following result.

Proposition 3. Let $S=I(u+\beta n)$ and $S'=I'(u+\beta' n')$ be two light beams crossing (in opposite directions) a system constituted by two ideal polarizers N and N'. The intensities of the outgoing light beams are related by

$$\frac{i(u)NN'(S')}{i(u)N'(S')}=\frac{i(u)N'N(S)}{i(u)N(S)}=\frac{1}{2}[1-(e,e')]. \tag{25}$$

A direct consequence of the last proposition is the following implication:

$$i(u)N(S)=i(u)N'(S')\Rightarrow i(u)NN'(S')=i(u)N'N(S)$$

that reflects the statement of the *law of reciprocity* in optics [8]: "If two incident polarized beams have equal intensities, the inverse emerging beams of the same polarization, which are associated with them, also have equal intensities." Now, this law follows as a corollary of Proposition 3.

Proposition 4 (reciprocity theorem). Let S and S' be two polarized beams and N and N' two ideal polarizers. If the intensity of $N(S)$ is equal to the intensity of $N'(S')$, then the outgoing beams $NN'(S')$ and $N'N(S)$ have also the same intensity.

Next, the results established above will be extended to the case of nonideal polarizers represented by Eq. (9). In this case, when an incoming light beam $S=I(u+\beta n)$ crosses A, the outgoing beam has the expression

$$A(S)=\frac{I}{2}\{[k_1+k_2-\beta(e,n)(k_1-k_2)]u+[k_1-k_2-\beta(e,n)\times(k_1+k_2)]e\}-\sqrt{k_1k_2}I\beta[(p,n)p+(q,n)q] \tag{26}$$

that reduces to Eq. (22) for $k_1=1$ and $k_2=0$. The u component of $A(S)$ will give the outgoing intensity, and then, we obtain the generalized Malus law.

Proposition 5 (generalized Malus law). When a light beam $S=I_{in}(u+\beta n)$ crosses a polarizer A represented by Eq. (9), the outgoing intensity is given by

$$i(u)A(S)=\frac{1}{2}[k_1+k_2-\beta(e,n)(k_1-k_2)]I_{in}. \tag{27}$$

In order to exhibit the Lorentzian character of the Malus law, let us express it in terms of the boost and homothetic parameter in the case of nonideal polarizers. Replacing Eq. (15) in Eq. (27), the Lorentz factor $1/\sqrt{1-b^2}$ emerges in a natural way. So, denoting $(e,n)=-\cos 2\theta$ according to our election of the Lorentzian signature, we obtain

$$I_{out}=\frac{a}{\sqrt{1-b^2}}(1+b\,\beta\cos 2\theta)I_{in}, \tag{28}$$

where $0<b<1$. As a consequence of Malus's law, the maximum and minimum transmitted intensities for incoming light of a given polarization degree β are given by

$$I_{\pm}=\frac{1}{2}[k_1+k_2\pm\beta(k_1-k_2)]I_{in}. \tag{29}$$

which allow one to define, for each β, a fractional polarization $P_A(\beta)$ associated with the polarizer,

$$P_A(\beta)=\frac{I_+(\beta)-I_-(\beta)}{I_+(\beta)+I_-(\beta)}=\beta\frac{k_1-k_2}{k_1+k_2}=\beta b. \tag{30}$$

This quantity is directly related to the polarizer boost parameter $b=P_A(1)$, that is, the polarization degree of the outgoing light when the incident one is natural, according to Eq. (14). Moreover, taking in mind the kinematical interpretation of the polarization degree given in Proposition 1, the aforementioned Lorentz factor, defined from the parameter b, inherites this kinematic meaning too. But, in general, this parameter is not equal to the polarization degree of the outgoing beam, as will be explained next. From Eq. (26) and using the decomposition (5) for the emerging beam, $A(S)=I_{out}(1,\vec{\beta}_{out})$, the outgoing normalized Stokes parameters are expressed as

$$\vec{\beta}_{out}=\frac{1}{1-b\beta(e,n)}\{[b+(\sqrt{1-b^2}-\beta)(e,n)]e+\sqrt{1-b^2}n\}, \tag{31}$$

where we have put

$$-[(p,n)p+(q,n)q]=n+(e,n)e.$$

Therefore, when $n\neq e$, $\vec{\beta}_{out}$ belongs to the two-plane generated by e and $\vec{\beta}_{in}=\beta n$. Otherwise, when $n=e$, $\vec{\beta}_{out}$, and $\vec{\beta}_{in}$ are collinear, and the polarization degree of the emerging beam obeys the relativistic law for the addition of two collinear "velocities" be and βe. Thus, the following property holds.

Proposition 6. When a light beam $S=I(u+\beta e)$ crosses a polarizer A represented by Eq. (9), the polarization degree of the outgoing light, $S_{out}=I_{out}(u+\beta_{out}e)$, is given by

$$\beta_{out}=\frac{b+\beta}{1+b\beta}. \tag{32}$$

There exist situations in Stokes space that are analogous to the "ultrarelativistic" limit of addition of velocities, that is, when we add the velocity v of a material particle and the light velocity in vacuum c this addition law gives, of course, c for every v. For example, in the case of Thomson scattering when $b=1$ at scattering angle $\theta=\pi/2$ and the scattered radiation is totally polarized ($\beta_{out}=1$).

Finally, we study the reciprocity relations for two general polarizers A and A', where A' is represented by Eq. (9) with the corresponding primed quantities k_1', k_2', e', p', q' ($\{u, e', p', q'\}$ being an orthonormal basis). Then, the intensity of a beam after crossing through A and A' is given by

$$i(u)A'A(S)=\frac{1}{2}[k_1'+k_2'-(k_1'-k_2')(e',\vec{\beta}_{A(S)})]i(u)A(S). \tag{33}$$

Note that the latter expression can also be directly obtained from the generalized Malus law (27) considering the outgoing intensity from the polarizer A' and the incoming beam $A(S)=I_{A(S)}(u+\vec{\beta}_{A(S)})$. Changing primed quantities for the corresponding unprimed ones, $A'(S')=I_{A'(S')}(u+\vec{\beta}_{A'(S')})$, and from Eq. (33) the corresponding expression for $i(u)AA'(S')$ may also be written. So, we obtain the following result using Proposition 5.

Proposition 7. Let $S=I(u+\beta n)$ and $S'=I'(u+\beta' n')$ be two light beams crossing (in opposite directions) an optical system of two polarizers A and A'. Then, the intensities of the outgoing beams satisfy

$$\frac{i(u)A'A(S)}{i(u)AA'(S')}=\frac{1-b'(e',\vec{\beta}_{A(S)})}{1-b(e,\vec{\beta}_{A'(S')})}\,\frac{1-b\beta(e,n)}{1-b'\beta'(e',n')}\,\frac{I}{I'}. \tag{34}$$

When the intensity of $A(S)$ is equal to the intensity of $A'(S')$ and for the particular case in which $n=e$ and $n'=e'$, the following simplified expression is obtained using Eq. (32):

$$i(u)A'A(S)=\frac{k_1'+k_2'}{k_1+k_2}\,\frac{1-\dfrac{b+\beta}{1+b\beta}b'(e,e')}{1-\dfrac{b'+\beta'}{1+b'\beta'}b(e,e')}\,i(u)AA'(S'). \tag{35}$$

The law of reciprocity for ideal polarizers (Proposition 4) is recovered taking $k_1=k_1'=1$, $k_2=k_2'=0$ and, accordingly Eq. (13), $b=b'=1$. For real polarizers the first fraction of the second term of Eq. (35) is related to the corresponding Lorentz factor. Effectively, from Eq. (15), we obtain in this case

$$\frac{k_1'+k_2'}{k_1+k_2}=\frac{a'}{a}\sqrt{\frac{1-b^2}{1-b'^2}}.$$

So, the Lorentz factors and the relativistic law of addition of velocities are present in the reciprocity relation given by Eq. (35). Indeed, a system of two real polarizers does not obey the reciprocity principle. According to Eq. (34), the emerging intensities of two polarized beams propagating in opposed directions are different even if their incoming intensities are equal. Therefore, Proposition 7 expresses the way in which realistic polarizers deviates from an exact reciprocity law behavior [18]. From a mathematical point of view this deviation has the same origin than the known relativistic effect named *Thomas precession*; the composition of two boosts along different directions A and A' depends on the order in which it is carried out [19]. In general, $AA'\neq A'A$ and they differ by an Euclidean rotation. This fact has been recently analyzed by several authors in polarization optics [20], but in a different context than the reciprocity relations considered here.

V. COMMENTS AND DISCUSSION

Stokes statements about the decomposition of natural and polarized light have been interpreted in a Minkowskian language using the properties of causal vectors in Lorentzian geometry and the "kinematic" interpretation of the polarization degree (Proposition 1). So, the Stokes statement [1,2]: "*Any partially polarized light may be regarded as the incoherent mixture of an unpolarized light and a completely polarized one*," reflects the property that any vector inside the positive shell of the null cone of a Lorentzian structure (positive oriented vector) may be decomposed as the sum of another positive oriented vector and a null vector with the same orientation [see Eq. (3)]. In this line, Eq. (8) refers to the incoherent decomposition of a partially polarized light as two totally polarized lights with opposite polarizations. Furthermore, the sum of positive and null vectors of the same orientation is always a positive vector with the given orientation, and so, any incoherent mixture of polarized light beams may be represented by a sole positive oriented Stokes vector.

On the other hand, the usual matrix representation of optical devices as polarizers and retarders also has a Lorentzian interpretation. Up to an overall factor, they can be seen as elements of the proper orthochronous subgroup of the Lorentz group acting on Stokes space. Matrices representing polarizers are homothetic to ordinary boosts, and retarders are represented by Euclidean rotations [17]. Moreover, as it has been shown in Sec. IV, Malus's law admits an elegant formulation in Stokes space (see Proposition 5) and allows us to interpret and extend the reciprocity relation for polarizers (Proposition 7). Note that the general expressions we have obtained for these laws can be specified in the special situations considered in Sec. III (the polarizers on reflection and refraction, and the Thomson scattering matrix) using in each case the given values of the transmission coefficients k_1 and k_2. Although we have considered these coefficients as constants it should be pointed out that, in general, they depend on the wavelength of the incident radiation. So, our results apply for each, but arbitrary, wavelength.

We must remark that the Stokes space formalism developed for polarization phenomena has one essential difference with the Minkowski space-time of special relativity, where future oriented timelike vectors represent equivalent inertial observers. In Stokes space, when a positive oriented vector is chosen to represent natural light, the other positive oriented vectors are interpreted as nonequivalent partially polarized lights.

It should be stressed that the Lorentzian structure used in this work, allows one to define the 2×2 Wolf coherency matrix W, whose trace and determinant are given by $\mathrm{tr}W=I$ and $4\det W=I^2-Q^2-U^2-V^2\equiv(S,S)\geqslant 0$. Then, Eqs. (4) and (7) for the polarization degree becomes $\beta^2 = 2\,\mathrm{tr}W^2/(\mathrm{tr}^2 W) - 1$, where we have considered the identity $\mathrm{tr}W^2 = \mathrm{tr}^2 W - 2\det W$. This equivalence between the Stokes and coherence matrix representations is clearly manifested in Fano's quantum description of polarized radiation [21]. In terms of the polarization degree, the eigenvalues $\lambda_\pm$ of W/I have the simple expression $\lambda_\pm = (1 \pm \beta)/2$. The entropy σ of the radiation field represented by W (or S) is $\sigma = -(\lambda_+ \ln \lambda_+ + \lambda_- \ln \lambda_-)$ and it can be expressed as a (nonlinear) function of β; then, the quantity $\psi(\beta) = [\ln(1+\beta) - \ln(1-\beta)]/2$ defines [4,22] an effective polarization temperature $\tau(\beta) = 1/\psi(\beta)$. For polarizers, the hyperbolic angle $\psi(b)$ given by Eq. (11) satisfies the above definition as a consequence of Eq. (15), and this allows us to introduce the notion of effective polarization temperature of a polarizer in a natural way, $\tau(b) = 1/\psi(b)$.

Finally, it has also to be remarked that Stokes formalism supposes implicitely that the light beam is uniformly polarized and is always propagated in the same direction. In order to characterize the spatial distribution of polarization over nonuniformly quasimonochromatic light beams, the Stokes formalism has been recently extended and an overall generalized polarization degree parameter has been introduced [23]. This parameter is defined in terms of the traces of the Stokes matrices which are subjected to a Lorentzian-type constraint formally similar to Eq. (1) [see Eqs. (14) and (16) in Ref. [23]]. Therefore, the Lorentzian interpretation may be in principle extended to deal with inhomogeneous spatial polarization distributions across light beams. On the other hand, in order to characterize the polarization properties of nonplane waves, Roman [24] developed a formalism based on the SU(3) expansion of the 3×3 coherency matrix R, whose coefficients give a generalized set of eigth normalized polarization parameters. The Lorentzian interpretation of Roman's formalism can also be done although, in this case, the structure to be considered has to be a Lorentzian polarization structure at each space-time point. Of course, in this case the generalized degrees of polarization are also related to the scalar invariants of R; the corresponding Lorentzian interpretation and the associated effective polarization temperatures claim our attention for further studies.

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